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The Aharonov-Bohm effect for a knotted magnetic solenoid

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Abstract

We show that the linking of a semiclassical path of a charged particle with a knotted magnetic solenoid results in the Aharonov-Bohm effect. The phase shift in the wave function is proportional to the flux intersecting a certain connected and orientable surface bounded by the knot (a Seifert surface of the knot).

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I. INTRODUCTION

The magnetic Aharonov-Bohm effect [1] results when a charged particle travels around a closed path in a region of vanishing magnetic field but nonvanishing vector potential. The magnetic flux is confined to a region where the particle is excluded, but the wave function of the particle is nonetheless affected by the vector potential and an interference pattern occurs at a detection screen.

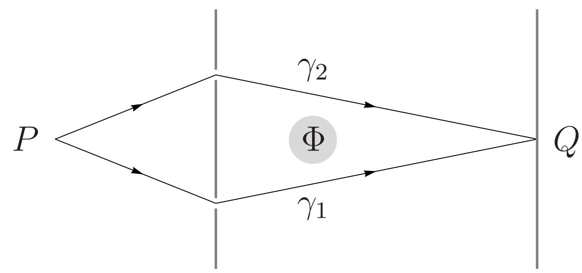

FIG. 1: A plane projection of the standard magnetic Aharonov-Bohm effect apparatus. The electrons travel along semiclassical paths γ_1 and γ_2 from the source P to the detection screen Q. The electrons do not penetrate the interior of the solenoid which carries the flux Φ.

Let C be the center line of a magnetic solenoid with the enclosed flux Φ and the vector potential $\boldsymbol{A}$. For an electron moving along a semiclassical path γ, the wave function is $\psi(\boldsymbol{A}) = \psi(0)\exp\left(i\xi\int_\gamma \boldsymbol{A}\cdot d\boldsymbol{x}\right)$, where $\xi = e(\hbar c)^{-1}$. For a standard magnetic Aharonov-Bohm effect apparatus in Fig. 1, the total amplitude for paths γ_1 and γ_2 is $\psi(\boldsymbol{A}) = e^{i\theta}(\psi_1(0)+ e^{-i\phi}\psi_2(0))$. The quantity θ is an overall irrelevant phase and the quantity $\phi = \xi\int_\gamma \boldsymbol{A}\cdot d\boldsymbol{x}$, where $\gamma = \gamma_1\gamma_2^{-1}$, measures the relative phase shift between paths γ_1 and γ_2. Applying the Stokes theorem, we find $\phi = \xi\Phi$.

There is a certain duality in this picture. Namely, if we take γ to be the center line of a solenoid with the magnetic flux Φ_* and C a path of an electron, then the phase shift is $\phi_* = \xi\Phi_*$. (We close the path C far away from the apparatus, e.g., at infinity.) The reason for this is the fact that the ratio of the phase to the flux is proportional to the gaussian linking of the curves γ and C, which is symmetric with respect to the curves. We will later use this duality in the more complicated case of knotted curves.

The phase ϕ is of course gauge invariant. For explicit computations, however, it is convenient to choose a singular gauge [2, 3, 4] in which $\boldsymbol{A} = \xi\Phi\delta_S\boldsymbol{n}_S$. Here S is a connected and orientable surface for which the curve C is the boundary, δ_S is the delta function with

the support on S, and $\boldsymbol{n}_S$ is the unit vector normal to S. (For an infinite solenoid, S is the half plane. For a toroidal solenoid, S is a disk.) It is clear that each time a closed path γ intersects the surface S, the quantity $\int_\gamma \boldsymbol{A} \cdot d\boldsymbol{x}$ increases or decreases by the quantity Φ depending on whether the intersection of γ and S is positive or negative. Thus, for an arbitrary path γ, the phase is $\phi = N\xi\Phi$, where the integer N is the signed number of times γ intersects S. In the dual picture, we consider a connected and orientable surface σ for which the curve γ is the boundary, choose the gauge potential $\boldsymbol{A}_* = \xi\Phi_*\delta_\sigma\boldsymbol{n}_\sigma$, and obtain $\phi_* = N_*\xi\Phi_*$, where N_* is the signed number of times C intersects σ. A crucial observation is that the two intersection numbers are equal, $N = N_*$.

For a fixed closed curve C, any possible path of an electron belongs to $C' = \mathbb{R}^3\backslash C$, the complement of C in $\mathbb{R}^3$. The set of all such paths form a group Γ under the operation of multiplication of paths. This group is $\Gamma = \pi_1(C')$, the first homotopy group [5] of C', also called the fundamental group of C'.

Topologically, C bounds a disk. If we continuously deform this disk, the signed intersection number does not change. This can be seen by noting that during deformations of S, new intersection points of γ and S appear in pairs, and the two points in each pair have intersections of opposite signs. This means that N depends only on the topological class to which $\gamma \in \Gamma$ belongs. It follows from the above that $\Gamma \cong \mathbb{Z}$. In the dual picture, we need $\Gamma_* = \pi_1(\mathbb{R}^3\backslash\gamma)$, the fundamental group of the complement of γ, and it similarly follows that $\Gamma_* \cong \mathbb{Z}$. In the next section we will generalize these ideas to orientable surfaces which are bounded by knots.

The above example has a simple topology, which resulted in abelian groups Γ and Γ_*. The Aharonov-Bohm analysis can be extended to examples of more complicated topologies. One possibility is to consider multiple magnetic solenoids, which might be unlinked or linked with each other. We have studied this case in Refs. [2, 3], where we showed that the phase is proportional to the product of fluxes from different solenoids and depends on linking numbers of higher orders. Our purpose here is to consider a simpler case of one self-knotted closed solenoid.

II. KNOTS

We now consider the case of a closed self-knotted curve C. For each such C, there are surfaces which are bounded by C. If such a surface is connected and orientable, then it is called a Seifert surface of C. A well-known theorem [6] states that there is a Seifert surface for every knot. (Note that some knots also bound non-orientable surfaces; for example, there is a Mobius strip with a 3π twist which is bounded by the trefoil knot. This is not a Seifert surface, and we will have no use for such non-orientable surfaces here.) In general, for a given C, there can be more than one nonequivalent Seifert surface [6, 7]. However, the signed number of intersections of a closed curve γ and S is independent of the choice of S. Hence our results will be independent of the choice of a Seifert surface. A Seifert surface for the trefoil knot is shown in Fig. 2.

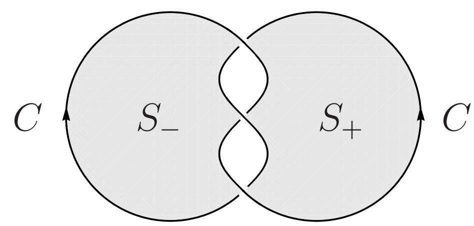

FIG. 2: The trefoil knot C and its Seifert surface S. S_+ and S_- are the two sides of the orientable S. If a semiclassical path of an electron intersects S, then the Aharonov-Bohm effect results.

A generalization of the Stokes theorem for knotted closed curves [8] states that $\int_\gamma \boldsymbol{A}\cdot d\boldsymbol{x} = N\Phi$, where N is the signed intersection number of γ and S, and S is a Seifert surface of the knot C. In the dual picture with the unknotted solenoid γ with flux Φ_* and a knotted semiclassical path of an electron C, we have $\int_C \boldsymbol{A}_* \cdot d\boldsymbol{x} = N_*\Phi_*$, where N_* is the signed intersection number of C and σ, and σ is a Seifert surface of γ. As in the case of unknotted closed curves, the numbers N and N_* are equal since they both again represent the gaussian linking of the curves γ and C.

For the fundamental group Γ with generators $\alpha_1, \ldots, \alpha_n$ satisfying relations $\beta_1 = 1, \ldots, \beta_n = 1$, we write [6]

$$\Gamma = (\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n). \tag{1}$$

The generators $\{\alpha_i\}$ and relators $\{\beta_i\}$ can be found for any knot C by using the Wirtinger presentation as follows. Let there be given a picture of the knot C as a set of arcs $C_1, \ldots, C_n$

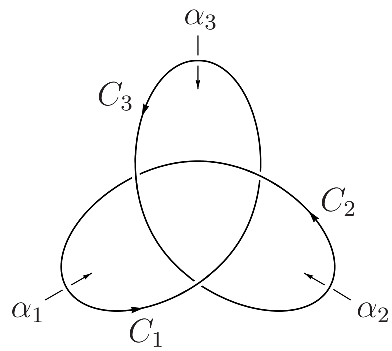

FIG. 3: A picture of the trefoil knot with the arcs C_1, C_2, C_3 and its Wirtinger presentation with the vectors α_1, α_2, α_3.

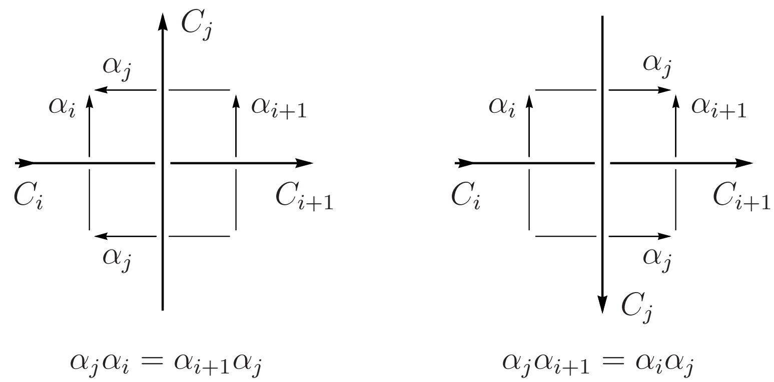

FIG. 4: The Wirtinger relations for a knot.

in a plane. Each C_i is assumed to be connected with C_{i-1} and C_{i+1} by undercrossing arcs as in Fig. 3. Indices are taken modulo n. We assume that the arcs are oriented in the order of $C_1, \dots, C_n$. For each arc C_i, we draw a vector α_i crossing C_i with the fixed orientation relative to the orientation of C_i. At each of the n crossings, there are two possibilities as in Fig. 4. These lead to the relations $\alpha_j\alpha_i = \alpha_{i+1}\alpha_j$ and $\alpha_j\alpha_{i+1} = \alpha_i\alpha_j$. It can be proved [6] that there are no other relations for the group Γ. This means that a relator β_i equals either $\alpha_j\alpha_i\alpha_j^{-1}\alpha_{i+1}^{-1}$ or $\alpha_j\alpha_{i+1}\alpha_j^{-1}\alpha_i^{-1}$. Since there is a product of all βs which is trivially the identity, any one of the relations defining Γ can be omitted.

Keeping the orientation and notation of Fig. 4, we can close the α lines to form circles. Then each element $\gamma \in \Gamma$ can be written in the form [9]

$$\gamma = \alpha_1^{k_{1,1}} \cdots \alpha_n^{k_{n,1}} \cdots \alpha_1^{k_{1,l}} \cdots \alpha_n^{k_{n,l}}, \tag{2}$$

where $l \in \mathbb{N}$, $k_{i,j} \in \mathbb{Z}$.

Since the first homotopy group $\Gamma = \pi_1(C')$ is now nonabilian and phases in quantum mechanics are elements of the abelian group U(1), we need to find $G = H_1(C')$, the first homology group of C', which is the abelianization of $\pi_1(C')$ [2, 3]. We obtain elements of G by considering elements of Γ modulo their commutators [5]. This means that we obtain G from Γ by replacing the non-commutative generators $\{\alpha_i\}$ and relators $\{\beta_i\}$ by commutative generators $\{a_i\}$ and relators $\{b_i\}$, respectively,

$$G = (a_1, \dots, a_n; b_1, \dots, b_n). \tag{3}$$

Here b_i equals either $a_i a_{i+1}^{-1}$ or $a_{i+1} a_i^{-1}$, which means that the corresponding relations allow us to replace a_{i+1} by a_i, or vice versa. As a result, the element $c \in G$ corresponding to $\gamma \in \Gamma$ is $c = a_h^m$, where h is any number from the set $\{1, \dots, n\}$ and $m = \sum_{i=1}^{n} \sum_{j=1}^{l} k_{i,j}$. Since $m \in \mathbb{Z}$, this implies that $G \cong \mathbb{Z}$ and a_h is the corresponding meridional generator.

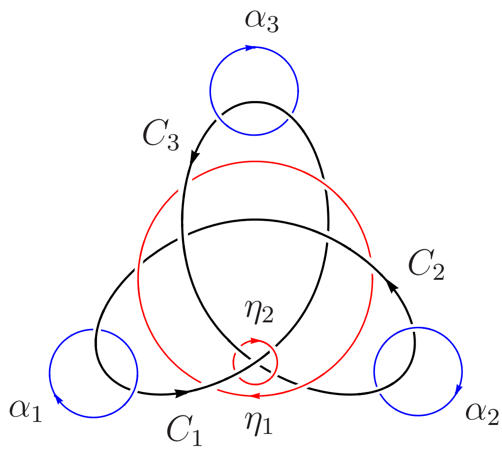

FIG. 5: The first homotopy group $\pi_1(T'_{2,3})$ is generated by the generators α_1, α_2, α_3 which satisfy the relations given by Eq. (5). Alternatively, $\pi_1(T'_{2,3})$ is generated by the generators η_1, η_2, which satisfy the relation $\eta_1^2 = \eta_2^3$.

Since G is an abelian group, the phase $\phi = N\xi\Phi$ is additive for multiplicative paths, as it must be in quantum mechanics [10], and now N is the number of times γ intersects the Seifert surface of the knot.

One of the simplest classes of nontrivial knots is the torus knots $\{T_{p,q}\}_{p,q\in\mathbb{Z}}$. The torus knot $T_{p,q}$ wraps around the solid torus in the longitudinal direction p times and in the meridional direction q times. We require that the numbers p and q are coprime and $|p| \neq 1$, $|q| \neq 1$, since otherwise C is unknotted. The simplest example in the family of torus knots is the trefoil knot $T_{2,3}$; see Fig. 3.

The simplest presentation of the fundamental group of $T'_{p,q} = \mathbb{R}^3 \backslash T_{p,q}$ is

$$\Gamma = (\eta_1, \eta_2; \eta_1^p \eta_2^{-q}). \tag{4}$$

For the trefoil knot, the Wirtinger relators are

$$\beta_1 = \alpha_3\alpha_2\alpha_3^{-1}\alpha_1^{-1}, \quad \beta_2 = \alpha_1\alpha_3\alpha_1^{-1}\alpha_2^{-1}, \quad \beta_3 = \alpha_2\alpha_1\alpha_2^{-1}\alpha_3^{-1}, \tag{5}$$

where the Wirtinger generators are related to the generators of $\pi_1(T'_{2,3})$ via $\eta_1 = \alpha_1\alpha_3\alpha_2$, $\eta_2 = \alpha_2\alpha_1$; see Fig. 5. Using the β_3 Wirtinger relation to eliminate α_3, we have $\eta_1 = \alpha_1\alpha_2\alpha_1$. It is now straightforward to check that $\eta_1^2 = \eta_2^3$ [11]. (The reader might find it useful to experiment with wires and strings to verify these results.) In general, the relations in the fundamental group preserve the signed intersection number of the closed path with the Seifert surface of the knot. For the trefoil knot example, see Fig. 6.

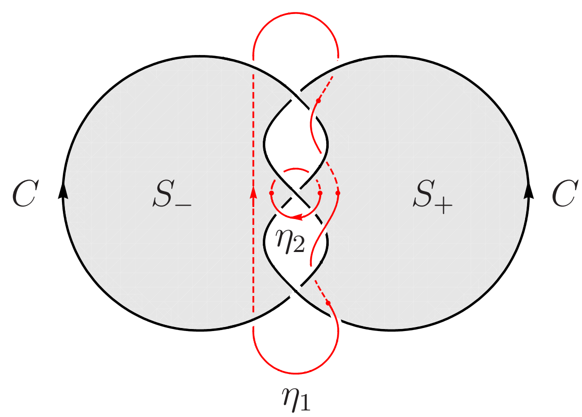

FIG. 6: An alternative view of the trefoil knot, where the generators η_1 and η_2 of $\pi_1(T'_{2,3})$ are shown intersecting the Seifert surface thrice and twice, respectively. The relation $\eta_1^2 = \eta_2^3$ implies that deforming η_1^2 into η_2^3 conserves the number of intersections of the path with the Seifert surface, which in this case is equal to six.

Note that there are paths, for example, γ in Fig. 7, that are linked with the knot, but that do not intersect the Seifert surface. Hence, if the path of an electron is linked through one of the holes in the Seifert surface, then there is no gaussian linking, and no standard Aharonov-Bohm effect. However, these paths have higher order linking with the knot and may result in a higher order effects where ϕ is nonlinear in flux as discussed in Refs. [2, 3]. The simplest example of higher order linking, the Borromean rings with an electron semiclassical path linking in a specific way with two unlinked solenoids, leads to a phase that is second order in fluxes [2].

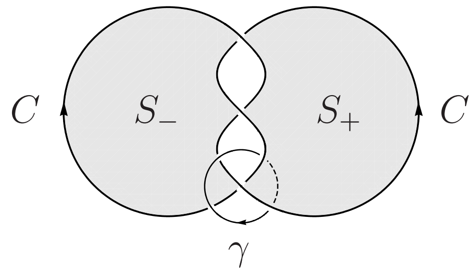

FIG. 7: An example of an electron path γ which does not intersect the Seifert surface S but nevertheless links with the knot C.

III. CONCLUSION

We conclude that the nontrivial phase that can be detected in the case of a knotted solenoid is $\phi = N(e\Phi)/(\hbar c)$, where N is the number of times the semiclassical path of an electron intersects a Seifert surface S of the knot C, and Φ is the magnetic flux within the knotted solenoid. This generalizes the case of a simple toroidal solenoid, but continues to correspond to the gaussian linking of the semiclassical particle path and the magnetic solenoid. Note that combining the methods developed here and in Refs. [2, 3], we in principle know the quantum-mechanical phase for the case of multiple knotted magnetic solenoids and a knotted path of an electron.

We will not suggest an experimental setup for detecting the Aharonov-Bohm effect for knots, since this is better left to those with the technical expertise who know best how to carry out such an experiment. However, it may be useful to approach the problem from the point of view of the Josephson effect where analogous generalized Aharonov-Bohm experiments [12] can be carried out.

Acknowledgments

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